Circular Lens Intersection Problem

Joseph G. Tylka
josephgt@alumni.princeton.edu

May 1^{st}, 2022

Summary

I came across this math problem on YouTube (see Fig. 1), and took a stab at solving it. After viewing the associated StackExchange post, I decided the answers there were missing an analytical expression, so I solve for it here. In it's most compact form, the solution is $(3\theta - 1)x^2$, where $x = 5$ cm and $\theta = \tan^{-1}\frac{1}{2}$.

1 Problem Formulation

The problem statement is the following: given the diagram in Fig. 1 below, solve for the area of the shaded region B. For a numerical solution, let $L = 10$ cm.

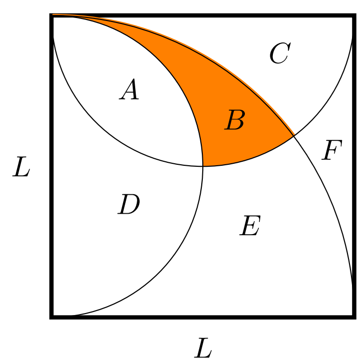

Figure 1: Diagram defining regions of interest, with the goal of solving for B.

2 Solution

We begin by identifying a few key regions to solve for, such as the circular lens comprising $A + B$. Later we will solve for this combined region via the 'kite' method, but first we solve geometrically for the area of A.

2.1 Computing A Geometrically

To solve for A, consider the following diagram in Fig. 2, where we have let $L = 2x$. From this, we see that the shaded regions 2Δ are equal to the total area of the square minus the area of the circle

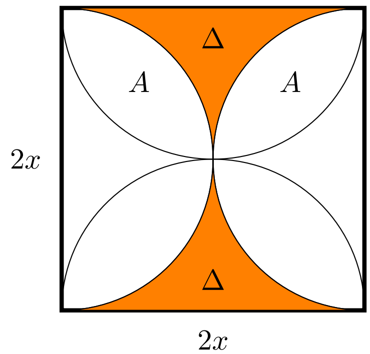

Figure 2: Diagram for computing Δ and A.

with radius x, i.e.:

$$2\Delta = (2x)^2 - \pi x^2 = (4-\pi)x^2 \implies \boxed{\Delta = \frac{4-\pi}{2}x^2} \tag{1}$$

Consequently, we can write the area of one semicircle with radius x as the following sum:

$$2A + \Delta = \frac{\pi}{2}x^2 \implies \boxed{A = \frac{\pi-2}{2}x^2} \tag{2}$$

2.2 Computing the Circular Lens

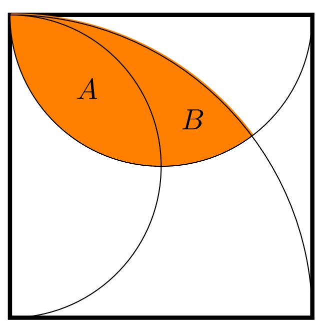

Figure 3: Diagram for computing the area of the circular lens, $A+B$.

Now we define the so-called *circular lens* of $A+B$ as shown in Fig. 3. To compute its area, we also draw the 'kite' in Fig. 4, which consists of multiple right triangles we can use. Note that here $\theta = \tan^{-1}\frac{1}{2}$ and $\theta + \phi = \frac{\pi}{2}$.

Recall that the area of a circular sector is given by $\alpha r^2/2$, where α is the span angle and r is the radius. So, we can compute the upper (cyan) and lower (pink) circular caps, κ_+ and κ_- by:

$$\kappa_+ = 4\theta x^2 - ab \quad \text{and} \quad \kappa_- = \phi x^2 - bc \tag{3}$$

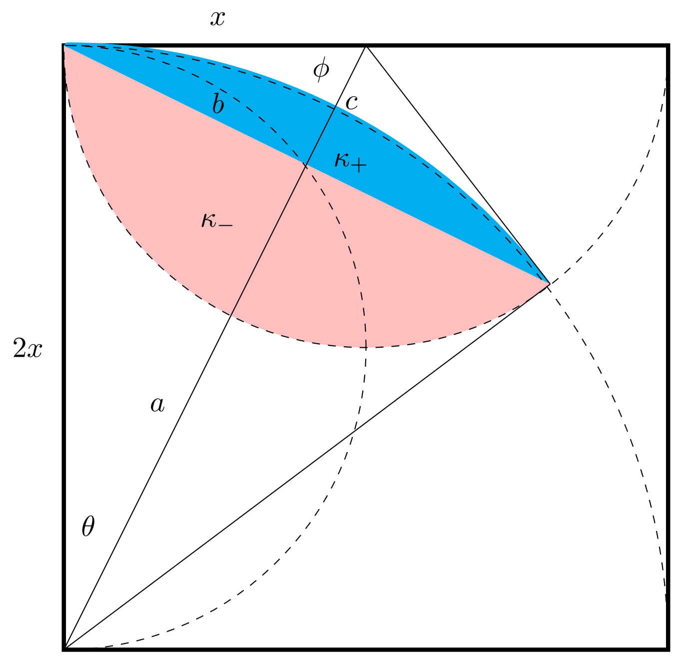

Figure 4: Diagram for computing the area of the circular lens, $\kappa_+ + \kappa_-$.

To simplify these expressions, note the trigonometric relationships defined by the right triangles of the kite:

$$a = 2x\cos\theta, \quad b = 2x\sin\theta = x\sin\phi, \quad c = x\cos\phi \tag{4}$$

$$\sin\theta = \cos\phi = \frac{1}{\sqrt{5}}, \quad \cos\theta = \sin\phi = \frac{2}{\sqrt{5}} \tag{5}$$

$$\implies \sin\theta\cos\theta = \sin\phi\cos\phi = \frac{2}{5} \tag{6}$$

Therefore,

$$ab = 4x^2\sin\theta\cos\theta = \frac{8}{5}x^2 \quad \text{and} \quad bc = x^2\sin\phi\cos\phi = \frac{2}{5}x^2 \tag{7}$$

So, now we can combine equations to find:

$$A + B = \kappa_+ + \kappa_- = 4\theta x^2 - ab + \phi x^2 - bc \tag{8}$$

$$= (4\theta + \phi)\,x^2 - x^2 = \left(3\theta + \frac{\pi}{2} - 2\right)x^2. \tag{9}$$

Therefore, subtracting A gives

$$\boxed{B = (3\theta - 1)x^2.} \tag{10}$$

For a numerical result, we substitute $\theta = \tan^{-1}(0.5)$ and $x = 5$ to obtain:

$$B = 75\tan^{-1}(0.5) - 25 \approx 9.774... \tag{11}$$

3 Extras

Just for completeness, let's find the rest of the areas shown in Fig. 1. Since we already know $A+B$, we can easily find C as:

$$C = \frac{\pi}{2}x^2 - A - B \implies \boxed{C = (2 - 3\theta)x^2} \tag{12}$$

To find D, we note that

$$A + D = \frac{\pi}{2}x^2 \implies \boxed{D = x^2} \tag{13}$$

Next, to find E, first note that

$$A + B + D + E = \frac{1}{4}\pi(2x)^2 = \pi x^2 \tag{14}$$

$$A + D = \frac{\pi}{2}x^2 \implies B + E = \frac{\pi}{2}x^2. \tag{15}$$

So, we can immediately write

$$\boxed{E = \left(\frac{\pi}{2} - 3\theta + 1\right)x^2} \tag{16}$$

Finally, to find F, note that

$$C + F = (4 - \pi)x^2 \implies \boxed{F = (2 - \pi + 3\theta)\, x^2} \tag{17}$$