Residues of Complex Functions Under a Linear Translation

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Abstract

Residues of a complex-valued function are shown to remain constant under a linear translation. This principle is then applied to simplify the process of computing an inverse Laplace transform via the Bromwich integral.

Linear Translation Invariance

Consider the complex-valued function, $f(z)$, with an m^{th} order pole at $z = z_0$. We can express $f(z)$ as a Laurent Series about the point $z = z_0$ with complex coefficients a_n, as shown below.

$$f(z) = \sum_{n=-m}^{+\infty} a_n (z - z_0)^n \tag{1}$$

The formula for calculating a residue of the function $f(z)$ at the point $z = z_0$ is given below.

$$\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \bigg|_{z=z_0} \tag{2}$$

We can add a linear translation to define a new variable $s = z + \gamma$ (and $z = s - \gamma$), where γ is a constant. We also define a new function, $g(s)$, such that $g(s) = f(s - \gamma)$, or, equivalently $g(z + \gamma) = f(z)$. We note that the pole of $f(z)$ at $z = z_0$ is also translated, so $g(s)$ has an m^{th} order pole at $s = z_0 + \gamma$. As before, we can express $g(s)$ as a Laurent Series, but now expanding about the singular point $s = z_0 + \gamma$ with complex coefficients b_n.

$$g(s) = \sum_{n=-m}^{+\infty} b_n (s - z_0 - \gamma)^n \tag{3}$$

As before, we can calculate the residue of the function $g(s)$ at the point $s = z_0 + \gamma$.

$$\operatorname*{Res}_{s=z_0+\gamma} g(s) = b_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} [(s - z_0 - \gamma)^m g(s)] \bigg|_{s=z_0+\gamma} \tag{4}$$

Since s is proportional to z and γ is a constant, we employ the chain rule to see that derivatives taken with respect to s are equal to those taken with respect to z, as shown in the pair of equations below.

$$\frac{d}{dz} = \frac{d}{ds}\frac{ds}{dz} = \frac{d}{ds}$$

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Hence, we may substitute $s = z + \gamma$ back into equation (4) to obtain the following expression:

$$\operatorname*{Res}_{z+\gamma=z_0+\gamma} g(z+\gamma) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z+\gamma-z_0-\gamma)^m g(z+\gamma)]\bigg|_{z+\gamma=z_0+\gamma}$$

The above equation is easily simplified to yield the same expression as equation (2), since $g(z+\gamma) = f(z)$.

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\bigg|_{z=z_0}$$

Thus, we have shown that the residues of a complex-valued function are constant with respect to a linear translation in the independent variable.

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{s=z_0+\gamma} g(s)} \tag{5}$$

This result can also be seen from equations (1) and (3). Since the two series are equivalent we see that $a_{-1} = b_{-1}$, which are the residues of the functions $f(z)$ and $g(s)$, respectively.

Application: Evaluating the Bromwich Integral

We can employ this result when calculating inverse Laplace transforms via the Bromwich integral, shown below.

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} X(s)e^{st}ds \tag{6}$$

γ is to the right of all the poles of $X(s)$

We define a new variable $s' = s - \gamma$ and a new function $\bar{X}(s') = X(s'+\gamma)$ and rewrite the Bromwich integral as shown below. Note that $ds' = ds$.

$$x(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \bar{X}(s')e^{(s'+\gamma)t}ds' \tag{7}$$

As is common practice in complex contour integration, we define a contour C_R as a semi-circle of radius R which encircles the negative-real half plane as $R \to \infty$. If it can be shown that the integral of $\bar{X}(s')e^{s't}$ on C_R approaches zero, for example via the M-L Bound or Jordan's Lemma, then by the Cauchy Residue Theorem, we may evaluate the Bromwich integral as a sum of residues. Note that the integral on C_R will only approach zero as R approaches infinity for non-negative values of time, $t \geq 0$, due to the $e^{s't}$ term. For negative values of time, we may choose C_R to encircle the positive-real half plane. The Cauchy Residue Theorem is shown below for some function $F(z)$ with n poles encircled by the contour C. Note the locations of the poles are $z = z_j$, for $j \in [1, n]$.

$$\oint_C F(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j}(F(z)) \tag{8}$$

As mentioned above, if the integral of $\bar{X}(s')e^{(s'+\gamma)t}$ around C_R goes to zero, then we can simply write the Bromwich integral from equation (7) as a sum of the residues of $\bar{X}(s')e^{(s'+\gamma)t}$.

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \bar{X}(s')e^{(s'+\gamma)t} ds' = \sum_{j=1}^{n} \operatorname*{Res}_{s'=s'_j} \left(\bar{X}(s')e^{(s'+\gamma)t} \right) \tag{9}$$

As demonstrated in the proof above, the residues of this new function, $\bar{X}(s')e^{(s'+\gamma)t}$, are equal to the residues of the original integrand $X(s)e^{st}$. Thus, by combining equation (9) with the result in equation (5), we see that the time domain function $x(t)$ is simply equal to the sum of the residues of $X(s)e^{st}$. Note that the locations of the residues of $\bar{X}(s')e^{(s'+\gamma)t}$ are $s' = s'_j$, and the locations of the residues of $X(s)e^{st}$ are $s = s_j = s'_j + \gamma$.

$$\boxed{x(t) = \sum_{j=1}^{n} \operatorname*{Res}_{s=s_j} \left(X(s)e^{st} \right)} \tag{10}$$

This result implies that the locations of the poles of $X(s)$ are irrelevant, provided the integral around the semi-circular contour C_R goes to zero, as discussed above.