Fourier Transform Identity for an Infinite Impulse Train

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Summary

A Fourier transform identity for an infinite Dirac delta impulse train is derived.

We begin by defining the Fourier transform and its corresponding inverse as

$$F(\omega) = \mathcal{F}\left[f(t)\right](\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt, \tag{1}$$

$$f(t) = \mathcal{F}^{-1}\left[F(\omega)\right](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega. \tag{2}$$

Now, consider an infinite train of Dirac delta impulses, given by

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT), \tag{3}$$

where T is the time-interval between pulses. We can directly evaluate the Fourier transform of $x(t)$ using the well-known integration property of the Dirac delta, given by

$$\int_{-\infty}^{+\infty} f(t)\delta(t - t_0) dt = f(t_0). \tag{4}$$

Thus, the Fourier transform of the infinite impulse train can be written as

$$X(\omega) = \sum_{n=-\infty}^{+\infty} e^{-j\omega n T}. \tag{5}$$

However, we note that $x(t)$ is periodic with period T. Consequently, we can express $x(t)$ as a Fourier series for periodic functions, given by

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi}{T}kt}, \tag{6}$$

where c_k are the Fourier coefficients of $x(t)$. These coefficients are found by taking the inner product of $x(t)$ with the k^{th} exponential over a given period, given by

$$c_k = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j\frac{2\pi}{T}kt} dt, \tag{7}$$

where t_0 is an arbitrary reference time. Substituting Eq. (3) yields

$$\begin{aligned} c_k &= \frac{1}{T}\int_{t_0}^{t_0+T} \sum_{n=-\infty}^{+\infty} \delta(t-nT)e^{-j\frac{2\pi}{T}kt}dt \\ &= \frac{1}{T}\int_{t_0}^{t_0+T} \delta(t-n_0T)e^{-j\frac{2\pi}{T}kt}dt \\ &= \frac{1}{T}e^{-j2\pi k n_0} = \frac{1}{T}, \end{aligned} \tag{8}$$

where k is an integer and n_0 is an integer chosen such that $t_0 < n_0T < t_0 + T$. Hence, the Fourier series representation of $x(t)$ is given by

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{j\frac{2\pi}{T}kt}. \tag{9}$$

The Fourier transform of this equation is then given by

$$X(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\frac{2\pi}{T}), \tag{10}$$

where this equality is obtained using the inverse Fourier transform of a Dirac delta in the frequency domain, given by

$$\mathcal{F}^{-1}\left[\delta(\omega-\omega_0)\right](t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega = \frac{e^{j\omega_0 t}}{2\pi}. \tag{11}$$

Therefore, we have shown that the Fourier transform of an infinite train of Dirac delta impulses can itself be written as an infinite train of Dirac delta impulses, given by

$$\boxed{\mathcal{F}\left[\sum_{n=-\infty}^{+\infty} \delta(t-nT)\right](\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\frac{2\pi}{T}).} \tag{12}$$